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Multivariate risk aversion utility, application to ESG investments

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ABSTRACT

This paper uses the concept of multivariate or multi-attributive utility to attach different risk aversion levels to different sources of wealth (e.g. sectors, stocks, asset classes). In this context, we address the topic of environmental, social, and corporate governance (ESG) investments from the perspective of an investor with different risk aversion levels to green and brown stocks. We obtain closed-form solutions for the optimal allocations, value function, and wealth equivalent losses (WEL) from suboptimal choices. The numerical analysis demonstrates the significant increase, of up to 33%, in green investments when accounting for a differential in risk aversions levels, with up to 65% in WEL when using same risk-aversion levels.[1]

1. Introduction

The foundations of Expected Utility Theory (EUT) has been around since the seminal work of Von Neumann and Morgenstern in the early 50 s, Neumann and Morgenstern (1953). The theory quickly became one of the main branches of portfolio optimization. Its popularity was cemented by the intuitive solutions derived in continuous-time by Merton in the 70 s for a Geometric Brownian motion using one of the broadest families of utility functions, the Hyperbolic Absolute Risk Aversion (HARA) utility, Merton (1971).

We use a family of multivariate utilities along the lines of the Cobb–Douglas Utility Function (see section 3.4 in Rasmussen (2011), and Campi and Owen (2011)) in the context of EUT, we call it the multivariate constant relative risk aversion (M-CRRA). We argue that investors might not consider or treat all sources of risk in their wealth equally in terms of relative risk-aversion levels. Some sources of risk in the market, for instance sectors or asset classes within the portfolio, could be of higher concern, dislike or lesser satisfaction than others; hence investors should have the flexibility to attach the right risk-aversion level to any given source of risk within their wealth, see for instance the recent work of Conine, McDonald, and Tamarkin (2017) Table 1 for empirical risk aversion levels on various asset classes. The idea of different sources and levels of felicity within a utility has also been studied under the keyword of multi-attributive utility functions, see Kihlstrom and Mirman (1974) and Dorfleitner and Krapp (2007); in this context, we can think of the risk on an asset class (or group of stocks) as an attribute.

Although our assumption is simple, the multivariate utility could also be interpreted as a merging of, on the one hand, investor's preferences over classes of assets, on the other hand, her risk preferences. These are two different concepts, but risk preferences have overshadowed asset-classes preferences in utility theory; rendering the latter to a mere exercise of including/excluding asset classes in the analysis, which is usually not ideal for investors. An interesting example of combining asset classes and risk preferences, or treating groups of assets with different levels of felicity, is the general topic of environmental, social, and corporate governance (ESG) investments. In the last 20 years, ESG driven investing has grown to more than US$30 trillion in assets under management, *Global sustainable investing assets surged to $30 trillion in 2018* (2019), with a clear upward grow trend in the future. For ESG purposes,

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corporations are evaluated on three fronts, first their work toward environmental goals, second their support of certain social movements, and thirdly whether the corporation is governed consistently with the diversity, equity, and inclusion movement, ESG (Environmental, Social and Governance) (0000). Many approaches have been designed to measure the alignment of corporations to ESG goals, this is called "ESG scores", see Whelan (2021) for a review of over 1,000 studies of different scores based on multiple data providers. Besides the corporations/firms/stocks "ESG scores", we should also consider the investor "ESG taste" (asset class preference, attribute preference), this can be interpreted as the investor's like or dislike for the ESG scores of the stocks in her portfolio.

Our goal in this paper is to include the ESG taste of an investor in a portfolio optimization analysis via risk aversion levels. We acknowledge that in the ESG literature, these two topics, asset class preference and risk aversion, have been treated separately, see Cornell (2021) and Pástor, Stambaugh, and Taylor (2021). We offer a simpler alternative with closed-form solutions and potential for generalizations. For simplicity, we classify stocks as either green or brown (two attributes with different level of felicity to investors), hence two possible risk-aversion levels. The term "green" means the firm generates positive externalities for society (i.e. it has a good ESG score), while "brown" impose negative externalities, i.e. low ESG score. The results in this paper can accommodate a more flexible relation between ESG score and risk-aversion level.

We can identify two general lines of research to ESG portfolio optimization in the literature. One relies on EUT, while the other line works with mean–variance theory (MVT). Given the novelty and the ongoing evolution of the topic of ESG portfolio optimization, we quickly mention some current works.

In the EUT branch, authors assumes that agents, besides liking wealth, derive utility from holding green stocks and disutility from holding brown stocks. An example of this line, in discrete time, is Pástor et al. (2021), where the authors consider N stocks (firms) and an exponential utility of the form $u(x) = -\exp\left(-\gamma x - b'\pi\right)$. Here x represents wealth, γ is the absolute risk aversion, π is a vector with the percentage of wealth allocated to risky assets, and b is a vector of nonpecuniary benefits that the agent derives from her stock holdings. The benefit vector has agent-specific and firm-specific components $b = dg$, where g is a firm-specific (i.e. ESG score) $N \times 1$ vector, and $d \geq 0$ is an agent-specific scalar measuring the degree of the agent's ESG taste. In a similar direction, Dorfleitner and Nguyen (2017) works with a mixed utility function: $u(x,s) = (1-\alpha)u_F(x) + \alpha u_S(s)$, where α in [0,1] can be interpreted as the general importance of the ESG goals of the investor (i.e. ESG taste) relatively to the financial aspects, x is wealth and s is the ESG return (i.e. ESG score), computed as $s = x_0\left(\sum \pi_i\left(1+s_i\right) - 1\right)$, where s_i is the ESG return of company i. They considered exponential utilities, see Jessen (2012) for a similar approach.

As for MVT, Schmidt (2020) considers a problem with objective function: $\frac{1}{2}\lambda\sigma_p^2 - \mu_p + \gamma\delta'\pi$, where μ_p and σ_p are the mean and standard deviations of the portfolio. γ captures the importance of ESG for the investor (i.e. ESG taste), and δ is a vector with portfolio constituents' ESG scores. In Gasser, Rammerstorfer, and Weinmayer (2017), the authors work with a similar objective function: $-\beta\sigma_p^2 + \alpha\mu_p + \gamma\theta$, where $\theta = \sum \pi_i\theta_i$ denotes the social responsibility rating/score of the portfolio (and each asset's ESG score), and γ indicating the social responsibility preference parameter of an investor (i.e. ESG taste). In De Spiegeleer, Höcht, Jakubowski, Reyners, and Schoutens (2021), the authors treat ESG taste as a constraint in the optimization of the form $\gamma\pi \leq \gamma_p$, where γ is a vector consisting of ESG scores for the portfolio constituents. While, Pedersen, Fitzgibbons, and Pomorski (2021) obtains the ESG efficient frontier for the objective function $-\frac{1}{2}\lambda\sigma_p^2 + \mu_p + xf(s)$, where $f(s)$ is the ESG preference function, which depends on the average ESG scores among the risky asset positions, this is $f(s) = f\left(\frac{s'\pi}{1'\pi}\right)$.

The paper is organized as follows, Section 2 describes the mathematical setting and problem, the target utilities and the main results. Section 3 studies the solution and its implications numerically. Section 4 concludes, while the proofs are presented in the appendix.

2. Setting and theoretical results

For simplicity, let us assume two stocks, S_1 and S_2, with the following dynamics:

$$\begin{aligned}
\frac{dS_{1,t}}{S_{1,t}} &= \left(r + \lambda_1\sigma_1^2\right)dt + \sigma_1 dW_{1,t} \\
\frac{dS_{2,t}}{S_{2,t}} &= \left(r + \lambda_1\sigma_1\sigma_2\rho + \lambda_2\sigma_2^2\sqrt{1-\rho^2}\right)dt + \sigma_2\left(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}\right) \\
&= \left(r + \lambda_{22}\sigma_2^2\right)dt + \sigma_2 dW_{3,t}
\end{aligned} \tag{1}$$

Here, $W_{1,t}$ and $W_{2,t}$ are uncorrelated Brownian motion, $-1 < \rho < 1$, $\sigma_i > 0$, $i = 1,2$; $\lambda_1\sigma_1$ is the market price of risk (MPR) associate to the risk factor W_1 and $\lambda_2\sigma_2$ is the MPR corresponding to risk factor W_2.

We have the following relation connecting the MPR of W_3 with those of W_1 and W_2:

$$\lambda_{22}\sigma_2 = \lambda_1\sigma_1\rho + \lambda_2\sigma_2\sqrt{1-\rho^2}$$

We assume the investor has wealth X_t, and it allocates the proportion of wealth $\pi_{1,t}$ into $S_{1,t}$, with $\pi_{2,t}$ into $S_{2,t}$ and the rest, $\left(1-\pi_{1,t}-\pi_{2,t}\right)$, in a bank account B_t with constant interest rate r. The self-financing condition for the investor reads:

$$\begin{aligned}
\frac{dX_t}{X_t} &= \pi_{1,t}\frac{dS_{1,t}}{S_{1,t}} + \pi_{2,t}\frac{dS_{2,t}}{S_{2,t}} + \left(1-\pi_{1,t}-\pi_{2,t}\right)\frac{dB_t}{B_t} \\
&= \left(r + \pi_{1,t}\lambda_1\sigma_1^2 + \pi_{2,t}\left(\lambda_1\sigma_1\sigma_2\rho + \lambda_2\sigma_2^2\sqrt{1-\rho^2}\right)\right)dt \\
&\quad + \pi_{1,t}\sigma_1 dW_{1,t} + \pi_{2,t}\sigma_2\left(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}\right)
\end{aligned}$$

The investor shows different preferences, i.e. risk-aversion levels, according to the source of risk, i.e. $W_{1,t}$ compared to $W_{2,t}$. For instance, $W_{1,t}$ represent the risk of the green stock, S_1, for which the investor is less risk averse. On the other hand, $W_{2,t}$ is the non-spanned (uncorrelated) risk associated to the brown stock S_2, here the investor considers higher risk aversions level as a way of capturing her ESG taste.

Next, we separate these two sources of risk in the construction of the wealth process and the utility function:

$$\begin{aligned} d\log X_t &= r dt + d\log X_{1,t} + d\log X_{2,t} \\ d\log X_{1,t} &= \left(\pi_{1,t}\lambda_1\sigma_1^2 + \pi_{2,t}\lambda_1\sigma_1\sigma_2\rho - \frac{1}{2}\left(\pi_{1,t}\sigma_1 + \pi_{2,t}\sigma_2\rho\right)^2\right) dt + \left(\pi_{1,t}\sigma_1 + \pi_{2,t}\sigma_2\rho\right) dW_{1,t} \\ d\log X_{2,t} &= \left(\pi_{2,t}\lambda_2\sigma_2^2\sqrt{1-\rho^2} - \frac{1}{2}\pi_{2,t}^2\sigma_2^2\left(1-\rho^2\right)\right) dt + \pi_{2,t}\sigma_2\sqrt{1-\rho^2} dW_{2,t} \end{aligned}$$

In this setting, $\log X_{1,T}$ captures the evolution of wealth driven by the green investment, while $\log X_{2,T}$ represents wealth associated to the non-spanned brown investment, these X_i can be interpreted as attributes leading to different levels of satisfaction on the investor. In particular, we have:

$$\begin{aligned} \frac{dX_{1,t}}{X_{1,t}} &= \left(\pi_{1,t}\lambda_1\sigma_1^2 + \pi_{2,t}\lambda_1\sigma_1\sigma_2\rho\right) dt + \left(\pi_{1,t}\sigma_1 + \pi_{2,t}\sigma_2\rho\right) dW_{1,t} \\ &= \left(\pi_{1,t}\sigma_1 + \pi_{2,t}\sigma_2\rho\right)\left(\lambda_1\sigma_1 dt + dW_{1,t}\right) \\ \frac{dX_{2,t}}{X_{2,t}} &= \pi_{2,t}\lambda_2\sigma_2^2\sqrt{1-\rho^2} dt + \pi_{2,t}\sigma_2\sqrt{1-\rho^2} dW_{2,t} \\ &= \pi_{2,t}\sigma_2\sqrt{1-\rho^2}\left(\lambda_2\sigma_2 dt + dW_{2,t}\right) \end{aligned}$$

with the relations:

$$\begin{aligned} \log\frac{X_T}{X_0} &= rT + \log\frac{X_{1,T}}{X_{1,0}} + \log\frac{X_{2,T}}{X_{2,0}} \\ X_0 &= X_{1,0}X_{2,0} \\ X_T &= \exp(rT) X_{1,T}X_{2,T} \end{aligned}$$

In order to differentiate the investor risk preferences, we use the following multiple constant relative risk aversion (M-CRRA) utility:

$$u(X_{1,T}, X_{2,T}) = sign(\alpha_1)\frac{\left(X_{1,T}\right)^{\alpha_1}}{\alpha_1}\frac{\left(X_{2,T}\right)^{\alpha_2}}{\alpha_2}$$

with $0 < \alpha_2 \leq \alpha_1 < 1$, or $\alpha_2 \leq \alpha_1 < 0$. Note $u(X_{1,T}, X_{2,T})$ is concave (Hessian is negative semi-definite), and increasing in each variable, while the Arrow–Pratt coefficient of absolute risk aversions depend on the variable/attribute:[2]

$$-\frac{\frac{\partial^2 u}{\partial(X_1)^2}}{\frac{\partial u}{\partial X_1}} = \frac{1-\alpha_1}{X_1}, \quad -\frac{\frac{\partial^2 u}{\partial(X_2)^2}}{\frac{\partial u}{\partial X_2}} = \frac{1-\alpha_2}{X_2}$$

The value function for this problem would be:

$$V\left(X_{1,0}, X_{2,0}\right) = \max_{\pi_{1,t},\pi_{2,t}} E\left[u(X_{1,T}, X_{2,T})\right] = \max_{\pi_{1,t},\pi_{2,t}} E\left[sign(\alpha_1)\frac{\left(X_{1,T}\right)^{\alpha_1}}{\alpha_1}\frac{\left(X_{2,T}\right)^{\alpha_2}}{\alpha_2}\right] \tag{2}$$

where, without loss of generality, we select initial values $X_{1,0} = X_0$, $X_{2,0} = 1$. Next, we present the main result of the paper.

Proposition 1. *Assuming $0 < \alpha_2 \leq \alpha_1 < 1$, or $\alpha_2 \leq \alpha_1 < 0$, the optimal allocations and value function for the problem in Eq. (2) are:*

$$\begin{aligned} \pi_2^* &= \frac{\lambda_2}{\left(1-\alpha_2\right)\sqrt{1-\rho^2}} \\ \pi_1^* &= \frac{\lambda_1}{\left(1-\alpha_1\right)} - \frac{\lambda_2\sigma_2\rho}{\sigma_1\left(1-\alpha_2\right)\sqrt{1-\rho^2}} \end{aligned}$$

$$\begin{aligned} V\left(X_{1,t}, X_{2,t}\right) &= sign(\alpha_1)\frac{X_t^{\alpha_1}}{\alpha_1\alpha_2}\exp\left(b(T-t)\right) \\ b &= \frac{1}{2}\left(\frac{\sigma_1^2\lambda_1^2\alpha_1}{\left(1-\alpha_1\right)} + \frac{\sigma_2^2\lambda_2^2\alpha_2}{\left(1-\alpha_2\right)}\right) \end{aligned}$$

[2] If one of the α is negative while the other is positive then the function is not concave. This multivariate utility is more general than Example 3.1 in Campi and Owen (2011); it satisfies assumptions 1.1 to 1.3, and definition 2.9 in Campi, Jouini, and Porte (2013); and it defines a KM risk averse investor (KM stands for Kihlstrom and Mirman in Kihlstrom and Mirman (1974)), also partial risk-averse with respect to each variable/attribute, as per theorem 1 in Dorfleitner and Krapp (2007)

See the proof in Appendix

Remark 2. We highlight next a few important aspects of this result and setting.

1. If ρ approaches 1 (−1), π_2 goes to ∞ and π_1 to $-\infty$ (∞). This is the same as in the case $\alpha_1 = \alpha_2$. The rationale is that, in the limit, one can create a risk free portfolio better than the cash account. This explains the assumption of invertible covariance matrix.
2. The work can be extended to N assets with the risk aversion structures $0 < \alpha_N \le \cdots \le \alpha_1 < 1$ or $\alpha_N \le \cdots \le \alpha_1 < 0$, and utility:
$$u(X_{1,T}, \dots, X_{N,T}) = \begin{cases} sign(\alpha_1) \prod_{i=1}^{N} \frac{\left(X_{i,T}\right)^{\alpha_i}}{\alpha_i} & N \text{ even} \\ \prod_{i=1}^{N} \frac{\left(X_{i,T}\right)^{\alpha_i}}{\alpha_i} & N \text{ odd} \end{cases}$$
A lower triangular representation for the stocks would be the most convenient to separate the independent risk factors (W_i, $i = 1, \dots, N$) driving the attributes (denoted X_i, $i = 1, \dots, N$), this is:
$$\frac{dS_{i,t}}{S_{i,t}} = \left(r + \sum_{j=1}^{i} \sigma_i \lambda_j \sigma_j \rho_{ij}\right) dt + \sigma_i \left(\sum_{j=1}^{i} \rho_{ij} dW_{j,t}\right)$$
In matrix for, $dS_t = diag(S_t)\left((r + diag(\sigma)diag(\lambda)A) + diag(\sigma)AdW_t\right)$, where S_t, W_t, λ and σ are vectors, and A represents a lower triangular decomposition of the correlation matrix $\rho = A'A$. This leads to a convenient representation of the self-financing condition:
$$\begin{aligned} \frac{dX_t}{X_t} &= \sum_{i=1}^{N} \pi_{i,t} \frac{dS_{i,t}}{S_{i,t}} + \left(1 - \sum_{i=1}^{N} \pi_{i,t}\right) \frac{dB_t}{B_t} \\ &= \sum_{i=1}^{N} \pi_{i,t} \left(\left(r + \sum_{j=1}^{i} \sigma_i \lambda_j \sigma_j \rho_{ij}\right) dt + \sigma_i \left(\sum_{j=1}^{i} \rho_{ij} dW_{j,t}\right)\right) + \left(1 - \sum_{i=1}^{N} \pi_{i,t}\right) \frac{dB_t}{B_t} \\ &= \sum_{i=1}^{N} \pi_{i,t} \sum_{j=1}^{i} \sigma_i \rho_{ij} \left(\lambda_j \sigma_j dt + dW_{j,t}\right) + \frac{dB_t}{B_t} \end{aligned}$$
Using $d \log X_t = rdt + \sum_{j=1}^{N} d \log X_{j,t}$ leads to a representation on independent wealths (attributes) X_j, $j = 1, \dots, N$:
$$\frac{dX_{j,t}}{X_{j,t}} = \bar{\pi}_j \left(\lambda_j \sigma_j dt + dW_{j,t}\right)$$
with $\bar{\pi}_j = \sum_{i=j}^{N} \pi_{i,t} \sigma_i \rho_{ij}$.
The problem can now be easily solved in terms of $\bar{\pi} = (\bar{\pi}_1, \dots, \bar{\pi}_N)'$ as $\bar{\pi}_j = \frac{\lambda_j}{(1-\alpha_j)\sigma_j}$, $j = 1, \dots, N$, and then transforming back to π via matrix notation $\bar{\pi} = A' diag(\sigma)\pi$.
3. In the scenario of two groups of stocks: $S_1, \dots, S_{N_1}$ and $S_{N_1+1}, \dots, S_N$, each with different risk aversions, α_A and α_B respectively, we can write using the notation of the previous remark:
$$\begin{aligned} d \log X_{A,t} &= \sum_{j=1}^{N_1} \log X_{j,t} \\ d \log X_{B,t} &= \sum_{j=N_1+1}^{N} \log X_{j,t} \\ d \log X_t &= rdt + d \log X_{A,t} + d \log X_{B,t} \end{aligned}$$
and the solution would follow similarly to the previous case by grouping $\alpha_1 = \cdots = \alpha_{N_1} = \alpha_A$, and $\alpha_{N_1+1} = \cdots = \alpha_N = \alpha_B$.
4. Interestingly, we could have written model (1) using the brown stock as the driver of the green stock, this is:
$$\begin{aligned} \frac{dS_{2,t}}{S_{2,t}} &= \left(r + \lambda_{22}\sigma_2^2\right) dt + \sigma_2 dW_{3,t} \\ \frac{dS_{1,t}}{S_{1,t}} &= \left(r + \lambda_{22}\sigma_1\sigma_2\rho + \lambda_{11}\sigma_1^2\sqrt{1-\rho^2}\right) dt + \sigma_1 \left(\rho dW_{3,t} + \sqrt{1-\rho^2} dW_{4,t}\right) \\ &= \left(r + \lambda_1\sigma_1^2\right) dt + \sigma_1 dW_{5,t} \end{aligned} \tag{3}$$
with the relation:
$$\lambda_1\sigma_1 = \lambda_{22}\sigma_2\rho + \lambda_{11}\sigma_1\sqrt{1-\rho^2}$$
The theory also goes through but with different solutions. The difference between the two models is how the investor interprets, in terms of risk aversion, the common/shared risk (correlated part) between the green and brown stock. (1) treats it with the lesser aversion (similar to the green stock), while (3) assumes it as per the brown (higher risk aversion).

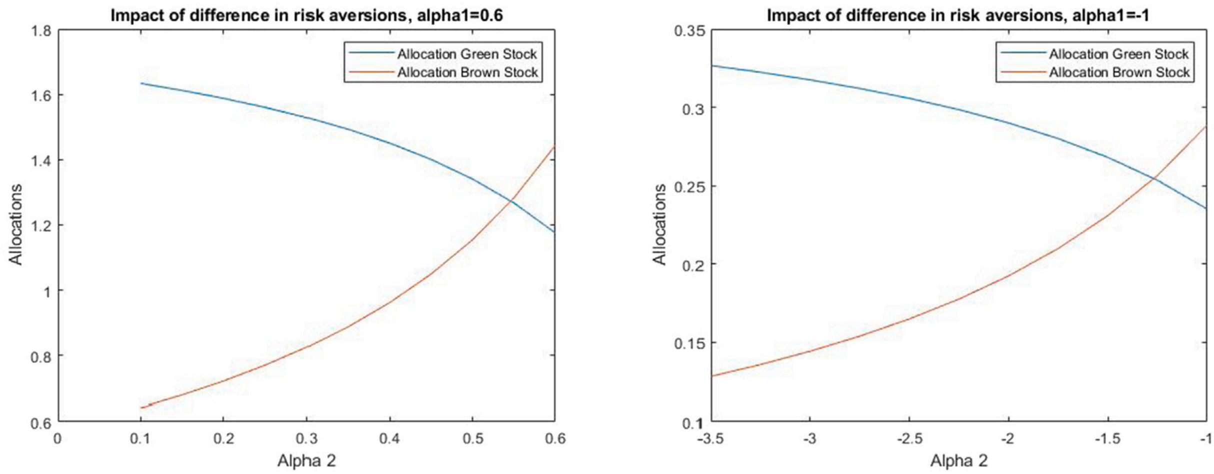

Fig. 1. Optimal allocation versus changes in α_2, left uses $\alpha_1 = 0.6$, right with $\alpha_1 = -1$.

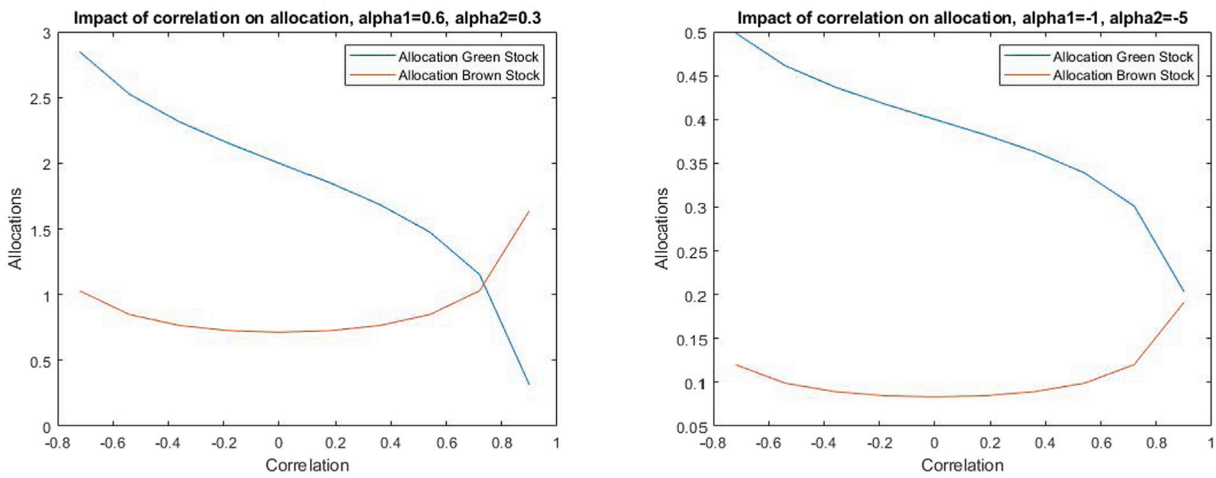

Fig. 2. Optimal allocation versus changes in correlation for, left uses $\alpha_1 = 0.6, \alpha_1 = 0.1$, right with $\alpha_1 = -1, \alpha_1 = -5$.

3. Numerical analysis and discussion

In this section we study the impact of different risk aversion levels and correlations on the optimal allocation, as well as the wealth equivalent losses incurred by using popular suboptimal strategies. We assume a standard annualized parametric setting with $\sigma_1 = 0.35$, $\sigma_2 = 0.4$, $\rho = 0.5$, $r = 0.01$, $\lambda_1 = 0.8$ and $\lambda_2 = 0.5$. This implies expected returns for assets 1 and 2 of $\mu_1 = 0.108$ and $\mu_2 = 0.1353$ respectively.

Fig. 1, left side, displays allocations in the stocks as a function of α_2, for a fix $\alpha_1 = 0.6$. The right side uses $\alpha_1 = -1$ and let α_2 go from -1 down to -5. In both cases, attaching a higher risk aversion to the brown stock leads to a significant increase in allocation to the green stock, from 1.2 to 1.6 (33% increase) in the left side of the figure, and from 0.25 to 0.33 on the right. As well as a large drop in brown stock investments, from 1.4 to 0.65 on the left (53% drop), while from 0.28 to 0.14 on the right side of the figure.

Fig. 2 displays the influence of correlation between the two stocks on the optimal allocations. On the left side, we fix $\alpha_1 = 0.6$ and $\alpha_2 = 0.1$, while on the right side $\alpha_1 = -1$, and $\alpha_2 = -5$. We can see that negative correlations lead to significantly higher allocation in green investments. This is particularly important given the notion that climate changes could create negative correlations between green stocks and brown stocks performances.

Next, we study the wealth equivalent losses (CEL) incurred by an investor who keeps the same risk-aversion levels due to lack of knowledge of how to construct her true optimal solution. The allocations obtained from using same risk aversion-level would be suboptimal, hence leading to a lost in utility. We denote the value function from a suboptimal strategy π as V^s, then the CEL is defined as the scalar q that satisfies the equation:

$$V(t, X_0(1-q), 1) = V^s(t, X_0, 1)$$

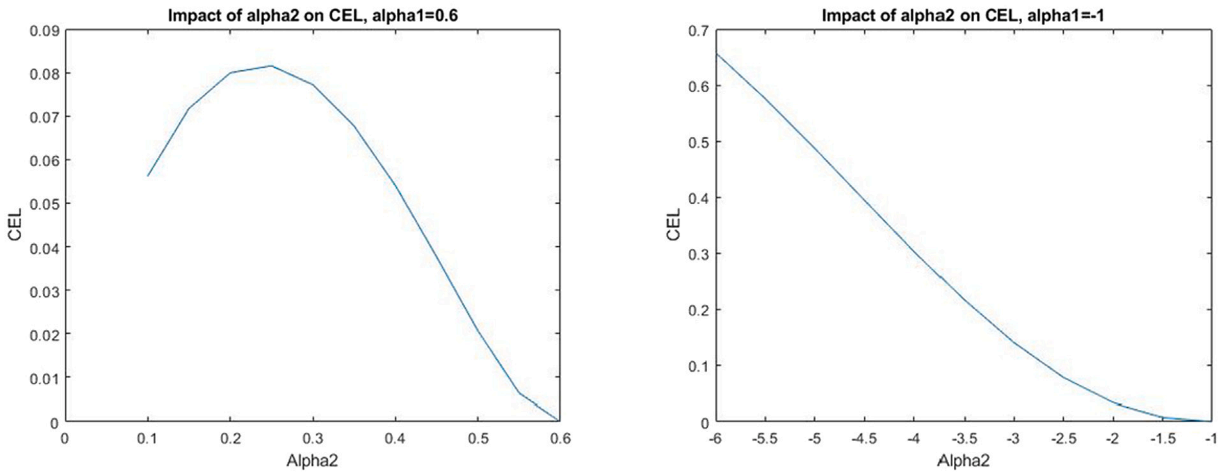

Fig. 3. Wealth Equivalent Loss versus changes in α_2, left uses $\alpha_1 = 0.6$, right with $\alpha_1 = -1$, the suboptimal uses $\alpha_2 = \alpha_1$.

We use Hamilton–Jacobi–Bellman equation with π_1 and pi_2 as the suboptimal, constant, strategies to obtain:

$$V^s(t,x,y) = sign(\alpha_1)\frac{X^{\alpha_1} v^s(t)}{\alpha_1\alpha_2}$$
$$v^s(t) = \exp\left(b^s\,(T-t)\right)$$

with

$$b^s = \left(\pi_1\lambda_1\sigma_1^2 + \pi_2\lambda_1\sigma_1\sigma_2\rho\right)\alpha_1 + \frac{1}{2}\left(\pi_1\sigma_1 + \pi_2\sigma_2\rho\right)^2\left(\alpha_1 - 1\right)\alpha_1$$
$$+\pi_2\lambda_2\sigma_2^2\sqrt{1-\rho^2}\alpha_2 + \frac{1}{2}\pi_2^2\sigma_2^2\left(1-\rho^2\right)\left(\alpha_2 - 1\right)\alpha_2$$

Then we find q as:

$$\frac{X^{\alpha_1}(1-q)^{\alpha_1}v(t)}{\alpha_1\alpha_2} = \frac{X^{\alpha_1}v^s(t)}{\alpha_1\alpha_2}$$
$$q = 1 - \left(\frac{v^s(t)}{v(t)}\right)^{1/\alpha_1}$$
$$q = 1 - \exp\left(\frac{1}{\alpha_1}\left(b^s - b\right)(T-t)\right)$$

Fig. 3 conveys that CEL losses due to staying with equal risk-aversion strategies could be of up to 65% of the initial investment. This can be seen in the right figure, where an investor who considers brown investments to be of $\alpha_2 = -6$ needs 65% less initial wealth to match the performance of an investor who also considers $\alpha_2 = -6$ to be the right risk-aversion level but uses the allocation implied by $\alpha_2 = -1$ instead.

4. Conclusions

This manuscripts uses the concept of multiple risk-aversions utilities, also known as multi-attributive utility, to open a new direction on ESG investments. Under the premise that investors would treat brown investment with higher risk-aversion than green investments, we obtain closed-form, intuitive, solutions to optimal allocations and value functions in an expected utility setting. This allows us to explore the implication of two risk-aversion settings in ESG investments for a reasonable choice of stock parameters.

The study can be extended in multiple directions, not only considering other multivariate utilities, multiple assets per risk aversion and richer, more realistic models to describe the underlyings, but also extending the ESG applications to other proposed forms of accounting for ESG taste and scores.

CRediT authorship contribution statement

Marcos Escobar-Anel: Conceptualization, Methodology, Software, Formal analysis, Data curation, Writing – original draft, Visualization, Investigation, Validation, Writing – review & editing.

Appendix. Proofs

Proof. Proof of 1. We can use Hamilton–Jacobi–Bellman equation (taking $x = x_1$, $y = x_2$ for simplicity):

$$0 = \inf_{\pi} \left\{ \begin{array}{c} V_t + \left(\pi_1\lambda_1\sigma_1^2 + \pi_2\lambda_1\sigma_1\sigma_2\rho\right) xV_x + \frac{1}{2}\left(\pi_1\sigma_1 + \pi_2\sigma_2\rho\right)^2 x^2 V_{xx} \\ +\pi_2\lambda_2\sigma_2^2\sqrt{1-\rho^2}\,yV_y + \frac{1}{2}\pi_2^2\sigma_2^2\left(1-\rho^2\right) y^2 V_{yy} \end{array} \right\}$$

$$V(T, x, y) = u(x, y) = sign(\alpha_1)\frac{x^{\alpha_1}}{\alpha_1}\frac{y^{\alpha_2}}{\alpha_2}$$

In this case the optimal proportions are:

$$\pi_1 = -\frac{\lambda_1 V_x}{xV_{xx}} - \pi_2\frac{\sigma_2\rho}{\sigma_1}$$

$$\pi_2 = \frac{-\lambda_1\sigma_1\rho xV_x - \lambda_2\sigma_2\sqrt{1-\rho^2}\,yV_y}{\sigma_2\left(\rho^2x^2V_{xx} + \left(1-\rho^2\right)y^2V_{yy}\right)} - \pi_1\frac{\sigma_1\rho x^2 V_{xx}}{\sigma_2\left(\rho^2x^2V_{xx} + \left(1-\rho^2\right)y^2V_{yy}\right)}$$

This can be simplified to:

$$\pi_2 = \frac{-\lambda_2 V_y}{\sqrt{\left(1-\rho^2\right)}\,yV_{yy}}$$

$$\pi_1 = -\frac{\lambda_1 V_x}{xV_{xx}} + \frac{\sigma_2\rho\lambda_2 V_y}{\sigma_1\sqrt{\left(1-\rho^2\right)}\,yV_{yy}}$$

Substituting in the HJB,

$$\begin{aligned} 0 = V_t &+ \left(\left(-\frac{\lambda_1V_x}{xV_{xx}} + \frac{\sigma_2\rho\lambda_2V_y}{\sigma_1\sqrt{(1-\rho^2)}yV_{yy}}\right)\lambda_1\sigma_1^2 + \frac{-\lambda_2V_y}{\sqrt{(1-\rho^2)}yV_{yy}}\lambda_1\sigma_1\sigma_2\rho\right)xV_x \\ &+\frac{1}{2}\left(\left(-\frac{\lambda_1V_x}{xV_{xx}} + \frac{\sigma_2\rho\lambda_2V_y}{\sigma_1\sqrt{(1-\rho^2)}yV_{yy}}\right)\sigma_1 + \frac{-\lambda_2V_y}{\sqrt{(1-\rho^2)}yV_{yy}}\sigma_2\rho\right)^2 x^2V_{xx} \\ &+\frac{-\lambda_2V_y}{\sqrt{(1-\rho^2)}yV_{yy}}\lambda_2\sigma_2^2\sqrt{1-\rho^2}\,yV_y + \frac{1}{2}\left(\frac{-\lambda_2V_y}{\sqrt{(1-\rho^2)}yV_{yy}}\right)^2\sigma_2^2\left(1-\rho^2\right)y^2V_{yy} \end{aligned}$$

$$0 = V_t - \frac{1}{2}\frac{\sigma_1^2\lambda_1^2V_x^2}{V_{xx}} - \frac{1}{2}\frac{\sigma_2^2\lambda_2^2V_y^2}{V_{yy}}$$

Assuming $V(t, x, y) = sign(\alpha_1)\frac{x^{\alpha_1}}{\alpha_1}\frac{y^{\alpha_2}}{\alpha_2}v(t)$ leads to:

$$0 = v' - \frac{1}{2}\frac{\alpha_1\sigma_1^2\lambda_1^2}{\left(\alpha_1-1\right)}v - \frac{1}{2}\frac{\alpha_2\sigma_2^2\lambda_2^2}{\left(\alpha_2-1\right)}v$$

$$v'(t) = -bv(t)$$

$$b = \frac{1}{2}\left(\frac{\sigma_1^2\lambda_1^2\alpha_1}{\left(1-\alpha_1\right)} + \frac{\sigma_2^2\lambda_2^2\alpha_2}{\left(1-\alpha_2\right)}\right)$$

$$v(t) = \exp\left(\frac{1}{2}\left(\frac{\sigma_1^2\lambda_1^2\alpha_1}{\left(1-\alpha_1\right)} + \frac{\sigma_2^2\lambda_2^2\alpha_2}{\left(1-\alpha_2\right)}\right)(T-t)\right) \quad \square$$

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