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ORIGINAL RESEARCH PAPER

Optimal multidimensional reinsurance policies under a common shock dependency structure

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Abstract
In this paper, we consider an insurance company that is active in multiple dependent lines. We assume that the risk process in each line is a Cramér–Lundberg process. We use a *common shock* dependency structure to consider the possibility of simultaneous claims in different lines. According to a vector of reinsurance strategies, the insurer transfers some part of its risk to a reinsurance company. Our goal is to maximize our objective function (*expected discounted surplus level integrated over time*) using a dynamic programming method. The optimal objective function (value function) is characterized as the unique solution of the corresponding Hamilton–Jacobi–Bellman equation with some boundary conditions. Moreover, an algorithm is proposed to numerically obtain the optimal solution of the objective function, which corresponds to the optimal reinsurance strategies.

Keywords Cramér–Lundberg process · Common shock · Dynamic programming principle · Reinsurance

1 Introduction

Stochastic control is an important area of research which has many applications in insurance. In particular, stochastic control is widely used to control the risk processes of insurance companies based on their reinsurance strategies. A reinsurance strategy is used by insurance companies to transfer some part of their risk to another

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insurance company. An important challenge for an insurance company is to optimize its reinsurance strategies. One approach for solving this problem which is widely used in the literature, is minimizing the ruin probability with respect to the reinsurance strategy. This approach was first studied in Schmidli [13] that assumed a Cramér–Lundberg risk process and a proportional reinsurance strategy. The approach presented in Schmidli [13] was extended to the excess of loss reinsurance strategy by Hipp and Vogt [6]. The same problem as Hipp and Vogt [6] was considered by Schmidli [14] and Taksar and Markussen [15]; however, it was based on the assumption that the risk process was the diffusion process. More recently, Cani and Thonhauser [4] and Cani [3] used a different objective function to find the optimum reinsurance strategy. In these works, the expected discounted surplus level function introduced by Højgaard and Taksar [7] and Højgaard and Taksar [8] is used as the objective function. Furthermore, a variety of optimization techniques has also been studied in Beveridge, Dickson, and Wu [2], Meng and Siu [10], Azcue and Muler [1], Eisenberg and Schmidli [5], Tamturk and Utev [16], Tan,Wei,Wei, and Zhuang [17], Preischl and Thonhauser [11] and Salah and Garrido [12] to find the optimum reinsurance strategy.

However, all mentioned works only focus on insurance companies active in just one line of business where the insurance company has only one type of reinsurance strategy for all its risks. Nonetheless, in practice, most insurance companies are usually active in more than just one line of business. These lines could be dependent such that controlling each line individually will not yield a global optimum result. Recently, Masoumifard and Zokaei [9] used the survival probability as the objective function to find a vector of optimal dynamic reinsurance strategies for an insurance company that operates on multiple independent lines.

In this work, we consider a common shock dependency structure for modeling the surplus process of a reinsurance company. Similar to Cani and Thonhauser [4], the expected discounted surplus level integrated over time is considered as our objective function. Our aim is to maximize this objective function with respect to a vector of reinsurance strategies.

In Sect. 2, a risk model for the surplus process with the common shock dependency structure is presented. The main results are stated in Sect. 3. In Sect. 4, a numerical algorithm for finding the optimal reinsurance strategies and the value function is explained. Finally, Sect. 5 provides the concluding remarks.

2 Common shock model

Consider an insurance company that operates on n dependent insurance lines. In practice, claims can occur simultaneously in several lines. For example, a car accident can cause damage both to the car and the driver as well. Hence, we assume that there are m sources as such that an occurrence in each source causes a claim in one or several lines. Figure 1 shows an example with four sources and three lines. In this example, sources 1 and 2 each makes claims only in one line; however, source 3 produces a claim in lines 2 and 3 simultaneously, and source 4 causes a claim in

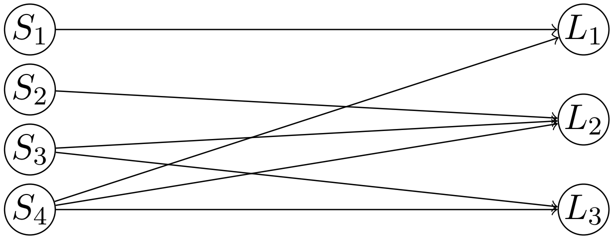


Fig. 1 An example of common shock model with $m = 4$ sources and $n = 3$ lines. $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{2, 3\}$ and $A_4 = \{1, 2, 3\}$

all lines. It should be noted sources 1 and 4 each makes a claim in the first line, but these claims can have different probability distributions.

More precisely, consider a probability space $(\Omega, \mathcal{E}, P)$. In this space, we consider independent Poisson processes $\{N_i(t) : t \geq 0\}$ with parameters β_i for the frequency of events in source $i \in \{1, 2, \dots m\}$. We denote the kth claim size from source i on line j, by the random variable Y_{ijk}. We assume $\{Y_{ijk} : i \in \{1, \dots, m\}, \quad j \in A_i, \quad k \in \mathbb{N}\}$ are independent random variables with cumulative distribution functions F^Y_{ij} and finite means μ_{ij}, where $A_i \subset \{1, 2, \dots, n\}$ is the set of lines affected by the source i. Therefore, the total amount of claims caused by the source i until time t is:

$$\sum_{k=1}^{N_i(t)} \sum_{j \in A_i} Y_{ijk} \qquad i = 1, \dots, m.$$

Given an initial capital x, then the surplus of the insurance company at time t is

$$X(t) = x + p\,t - \sum_{i=1}^{m} \sum_{k=1}^{N_i(t)} \sum_{j \in A_i} Y_{ijk}, \tag{1}$$

where, p is the premium rate and is calculated using the expected value principle with relative safety loadings $\eta_j > 0$ that is:

$$p = \sum_{j=1}^{n} p_j, \qquad p_j = (1 + \eta_j) \sum_{i=1}^{m} \beta_i \mu_{ij} I_{A_i}(j), \tag{2}$$

in which $I_A(x)$ is the indicator function.

2.1 Reinsurance

In this subsection, we define reinsurance strategies for a company with n lines of business. First consider the filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$, where $\mathcal{F}_t$ is the σ-algebra generated by $\{X(s) \;\; 0 \leq s \leq t\}$.

A reinsurance strategy is a multivariate stochastic processes $\boldsymbol{U} = \{\boldsymbol{U}(t) = (U_1(t), \dots, U_n(t)) : t \geq 0\}$. If at time $t = t_1$ we have a claim of size $Y = \sum_{j \in A_i} Y_{ijk}$ from the source i, then the reinsurance company covers $Y - \sum_{j \in A_i} r_j\big(U_j(t_1), Y_{ijk}\big)$, where the functions

$0 \le r_j(u, y) \le y$ are continuous and increasing in y. We say that $\boldsymbol{U}$ is admissible if for $j = 1, 2, \dots, n$, the functions $(\omega, t, y) \to r_j\big(U_j(\omega, t), y\big)$ are $\mathcal{E} \times \mathcal{B} \times \mathcal{B}$ measurable and functions $\omega \to \sum_{j \in A_i} r_j\big(U_j(\omega, t), y\big)$ are $\mathcal{F}_{t^-}$ measurable for every $t \ge 0$ and $y \ge 0$. We denote the set of all admissible strategies by $\boldsymbol{\mathcal{R}}$. In this paper, we consider each line can have one of the following reinsurance contracts:

(1) *Proportional*: $r^P(u, y) = uy, \quad u \in \mathcal{U}^P = [0, 1]$.
(2) *Excess of loss* (XL): $r^{XL}(u, y) = \min(u, y), \quad u \in \mathcal{U}^{XL} = [0, \infty]$.

Using (1), the surplus process controlled by reinsurance strategy $\boldsymbol{U}$ is given by

$$X_{\boldsymbol{U}}(t) = x + \int_0^t p(\boldsymbol{U}(s))\mathrm{d}s - \sum_{i=1}^{m} \sum_{k=1}^{N_i(t)} \sum_{j \in A_i} r_j\big(U_j(\tau_{ik}^-), Y_{ijk}\big), \tag{3}$$

where τ_{ik} is the time of the kth claim from the source i. We use the expected value principle for calculating the premium:

$$p(\boldsymbol{u}) = p - \sum_{j=1}^{n} q_j(u_j), \quad \boldsymbol{u} \in \mathcal{U}$$

and

$$q_j(u_j) = (1 + \theta_j) \sum_{i=1}^{m} \beta_i E\big(Y_{ijk} - r_j\big(u_j, Y_{ijk}\big)\big) I_{A_i}(j),$$

where, $\theta_j > \eta_j$ is the safety loading factor and p is defined in (2) and $\mathcal{U} \subset \mathbb{R}^n$ is the set $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$. For proportional lines we have $\mathcal{U}_j = [0, 1]$ and for excess of loss $\mathcal{U}_j = [0, \infty]$.

2.2 The value function

Given a reinsurance strategy $\boldsymbol{U}$ and an initial surplus $x \ge 0$, similar to Cani and Thonhauser [4], we define the following objective function:

$$\begin{aligned} V_{\boldsymbol{U}}(x) &= E\left(\int_0^{\tau_U} e^{-\delta s} X_{\boldsymbol{U}}(s)\mathrm{d}s \middle| X_{\boldsymbol{U}}(0) = x \right) \\ &= E_x\left(\int_0^{\tau_U} e^{-\delta s} X_{\boldsymbol{U}}(s)\mathrm{d}s \right), \end{aligned} \tag{4}$$

where $\tau_{\boldsymbol{U}}$ is the time of ruin

$$\tau_{\boldsymbol{U}} = \inf\big\{ t \ge 0 : X_{\boldsymbol{U}}(t) < 0 \big\},$$

and $\delta > 0$ is a discount rate. This function is known as *expected discounted surplus level integrated over time*. The value function is given by (5)

$$V(x) = \sup_{\boldsymbol{U}\in\mathcal{R}} V_{\boldsymbol{U}}(x), \tag{5}$$

and the optimal reinsurance strategy U^* is such that $V = V_{U^*}$.

Similar to Proposition 1, Lemmas 1 and 2 of Cani and Thonhauser [4] it is easy to see that the value function $V(x)$ is strictly increasing, locally Lipschitz continuous; therefore, absolutely continuous, and for all $x \geq 0$ we have

$$\frac{x}{\delta} < V(x) \leq \frac{x}{\delta} + \frac{p}{\delta^2}. \tag{6}$$

3 Hamilton-Jacobi-Bellman equation

To be able to solve (5) using methods of dynamic programming and to further ensure the optimality of the solution, we first need to find the Hamilton–Jacobi–Bellman equation associated to $V(x)$.

Lemma 1 *The value function $V(x)$ is a.e. a solution to:*

$$\sup_{\{\boldsymbol{u}\in\mathcal{U}|p(\boldsymbol{u})\geq 0\}} H_g(x,\boldsymbol{u}) = 0, \tag{7}$$

where

$$H_g(x,\boldsymbol{u}) = x + p(\boldsymbol{u})g'(x) - (\delta+\beta)g(x) + \sum_{i=1}^{m} \beta_i \int_0^x g(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z),$$

and $\beta = \sum_{i=1}^m \beta_i$ and $F_i^{\boldsymbol{u}}(z)$ is the cumulative distribution function of $\sum_{j\in A_i} r_j(u_j, Y_{ijk})$.

Proof Because the proof is similar to the proof of Lemma 3 of Cani and Thonhauser [4], we only identify the infinitesimal generator of (3) controlled by the constant strategy $\boldsymbol{u} = (u_1, \dots, u_n)$. Consider $Z_i = \sum_{j\in K_i} r(u_j, Y_{ij})$. Similar to page 10 of Azcue and Muler [1], define $B_0 = \{N_1(t) = 0, \dots, N_m(t) = 0\}$, $B_i = \{N_i(t) = 1, N_k(t) = 0 \quad \forall k \neq i\}$ for $i = 1, 2, \dots, m$ and $B_{m+1} = \left(\cup_{j=0}^m A_j\right)^c$, we have the following relation

$$\begin{aligned} E\big(g\big(X_{\boldsymbol{u}}(t)\big)\big) &= \sum_{j=0}^{m+1} E\Big(g\big(X_{\boldsymbol{u}}(t)\big)I_{B_j}\Big) \\ &= g(x+p(\boldsymbol{u})t)e^{-\beta t} + \sum_{j=1}^{m} E\Big(g\big(X_{\boldsymbol{u}}(t)\big)I_{B_j}\Big) + o(t), \end{aligned}$$

where I is the indicator function. Therefore

$$\lim_{t\to 0^+} \frac{g(x+p(\boldsymbol{u})t)e^{-\beta t} - g(x)}{t} = p(\boldsymbol{u})g'(x) - \beta g(x),$$

and

$$E\big(g\big(X_{\boldsymbol{u}}(t)\big)I_{B_i}\big) = \beta_i \int_0^t e^{-\beta_i s} \int_0^{x+p(\boldsymbol{u})s} g(x + p(\boldsymbol{u})s - z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z)\,\mathrm{d}s.$$

Therefore

$$\lim_{t\to 0^+} \frac{E\big(g\big(X_{\boldsymbol{u}}(t)\big)I_{B_i}\big)}{t} = \beta_i \int_0^x g(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z).$$

So the infinitesimal generator is

$$\mathcal{A}_{X_{\boldsymbol{u}}} g(x) = p(\boldsymbol{u})g'(x) - \beta g(x) + \sum_{i=1}^m \beta_i \int_0^x g(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z).$$

Now we can simplify $\sum_{i=1}^m \beta_i \int_0^x g(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z)$ as:

$$\sum_{i=1}^m \beta_i \int_0^x g(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z) = \beta \int_0^x g(x-z)\,\mathrm{d}F^{\boldsymbol{u}}(z),$$

where $F^{\boldsymbol{u}}(z) = \frac{1}{\beta}\sum_{i=1}^m \beta_i F_i^{\boldsymbol{u}}(z)$. Therefore $\mathcal{A}_{X_{\boldsymbol{u}}} g(x)$ is similar with the one in the proof of Lemma 3 of Cani and Thonhauser [4] and so the continuation of the proof is similar. □

Remark 1 Suppose that x is such that $V'(x)$ exists. Since $H_V(x, \boldsymbol{u})$ is continuous in $\boldsymbol{u}$ and $\{\boldsymbol{u} \in \mathcal{U} | p(\boldsymbol{u}) \geq 0\} \subset \mathbb{R}^n$ is compact, there exists the pointwise maximizer $\boldsymbol{u}^*(x) = (u_1^*(x), \dots, u_n^*(x))$ such that $H_V(x, \boldsymbol{u}^*(x)) = 0$. Therefore

$$p(\boldsymbol{u}^*(x))V'(x) = (\delta + \beta)V(x) - x - \sum_{i=1}^m \beta_i \int_0^x V(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}^*(x)}(z).$$

By (6), we have $V(x) > x/\delta$, and since $V(x)$ is increasing, we can write

$$(\delta + \beta)V(x) - x - \sum_{i=1}^m \beta_i \int_0^x V(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}^*(x)}(z) \geq \delta V(x) - x > 0.$$

Therefore $p(\boldsymbol{u}^*(x))V'(x) > 0$ which means $p(\boldsymbol{u}^*(x)) > 0$. So $V(x)$ is also an a.e. solution to

$$g'(x) = \mathcal{D}g(x), \tag{8}$$

where

$$\mathcal{D}g(x) = \inf_{\{\boldsymbol{u}\in\mathcal{U}|p(\boldsymbol{u})>0\}} \frac{(\delta+\beta)g(x) - x - \sum_{i=1}^m \beta_i \int_0^x g(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z)}{p(\boldsymbol{u})}.$$

Also, if $g(x)$ is absolutely continuous, (8) is equivalent to

$$g(x) = g(x_0) + \int_{x_0}^{x} \mathcal{D}g(z)\, \mathrm{d}z. \tag{9}$$

Lemma 2 *If $g_1(x)$ and $g_2(x)$ are two absolutely continuous solutions of* (8) *such that $g_1(0) \le g_2(0)$, then $h(x) = g_2(x) - g_1(x)$ is a non decreasing function.*

Proof Define

$$J_g(x, \boldsymbol{u}) = \frac{(\delta + \beta)g(x) - x - \sum_{i=1}^{m} \beta_i \int_0^x g(x - z)\, \mathrm{d}F_i^{\boldsymbol{u}}(z)}{p(\boldsymbol{u})}.$$

Note that $p(\boldsymbol{u}) \le p$, $J_{g_2}(x, \boldsymbol{u})$ is continuous in $\boldsymbol{u}$ and $\{\boldsymbol{u} \in \mathcal{U} | p(\boldsymbol{u}) \ge 0\}$ is compact. Assume that $\boldsymbol{u}^{(2)}(x) = (u_1^{(2)}(x), \ldots, u_n^{(2)}(x))$ is the pointwise minimizer for $g_2(x)$ i.e.

$$J_{g_2}(x, \boldsymbol{u}^{(2)}(x)) = \inf_{\{p(\boldsymbol{u})>0\}} J_{g_2}(x, \boldsymbol{u}) = \mathcal{D}g_2(x).$$

We have

$$\begin{aligned} \mathcal{D}g_2(x) - \mathcal{D}g_1(x) &\ge J_{g_2}(x, \boldsymbol{u}^{(2)}(x)) - J_{g_1}(x, \boldsymbol{u}^{(2)}(x)) \\ &= \frac{(\delta + \beta)h(x) - \sum_{i=1}^{m} \beta_i \int_0^x h(x - z)\, \mathrm{d}F_i^{\boldsymbol{u}^{(2)}(x)}(z)}{p(\boldsymbol{u}^{(2)}(x))} \\ &\ge \frac{(\delta + \beta)h(x) - \beta \sup_{z\in[0,x]} h(z)}{p}. \end{aligned} \tag{10}$$

Now by contradiction we prove that for all $x \ge 0$ we have $\sup_{z\in[0,x]} h(z) = h(x)$. Assume that there exists x, such that $\sup_{z\in[0,x]} h(z) > h(x)$. Since $h(x)$ is continuous, there exists $0 \le x_0 < x$, such that $\sup_{z\in[0,x]} h(z) = h(x_0) > h(0) \ge 0$. Since $h(x)$ is a continuous function and $h(x_0) > 0$, there exist $0 < \epsilon < x - x_0$ such that for $z \in [x_0, x_0 + \epsilon)$ we have $(\delta + \beta)h(z) - \beta h(x_0) > 0$. So by (10), for $z \in [x_0, x_0 + \epsilon)$ we have

$$\mathcal{D}g_2(z) - \mathcal{D}g_1(z) \ge \frac{(\delta + \beta)h(z) - \beta h(x_0)}{p} > 0.$$

But by (9) $h(x_0 + \epsilon) = h(x_0) + \int_{x_0}^{x_0+\epsilon} \left(\mathcal{D}g_2(z) - \mathcal{D}g_1(z)\right) \mathrm{d}z > h(x_0)$ which is a contradiction. This contradiction proves that for all $x \ge 0$, $\sup_{z\in[0,x]} h(z) = h(x)$. □

Lemma 3 *If $g_1(x)$ and $g_2(x)$ are two absolutely continuous solutions of* (8) *such that $g_1(0) = g_2(0)$, then for all $x \ge 0$ we have $g_1(x) = g_2(x)$.*

Proof From Lemma 2, we know that both $h(x) = g_2(x) - g_1(x)$ and $-h(x) = g_1(x) - g_2(x)$ are non decreasing. Therefore $h(x)$ is a constant function. So $h(x) = h(0) = 0$. □

In the next theorem we prove that $V(x)$ is the only solution of (8) which satisfies the inequality (6).

Theorem 1 *$V(x)$ is the unique absolutely continuous solution of (8) with $g(0) = V(0)$. If $g_1(x)$ and $g_2(x)$ are two absolutely continuous solutions of (8) such that $g_1(0) < V(0) < g_2(0)$, then there exists x_0 such that for $x > x_0$ we have $g_1(x) < \frac{x}{\delta}$ and $g_2(x) > \frac{x}{\delta} + \frac{p}{\delta^2}$.*

Proof By Lemma 1, $V(x)$ is a solution to (8) and by Lemma 3 it is unique. From Lemma 2 we know $h_1(x) = V(x) - g_1(x)$ and $h_2(x) = g_2(x) - V(x)$ are two non decreasing functions. By using (10) for all $x \geq 0$, we have

$$\mathcal{D}V(x) - \mathcal{D}g_1(x) \geq \frac{\delta h_1(x)}{p} \geq \frac{\delta h_1(0)}{p} > 0,$$

and

$$\mathcal{D}g_2(x) - \mathcal{D}V(x) \geq \frac{\delta h_2(x)}{p} \geq \frac{\delta h_2(0)}{p} > 0.$$

Therefore by (9)

$$h_1(x) = h_1(0) + \int_0^x \left[\mathcal{D}V(z) - \mathcal{D}g_1(z)\right] \mathrm{d}z \geq h_1(0) + \frac{\delta h_1(0)}{p} x,$$

and

$$h_2(x) = h_2(0) + \int_0^x \left[\mathcal{D}g_2(z) - \mathcal{D}V(z)\right] \mathrm{d}z \geq h_2(0) + \frac{\delta h_2(0)}{p} x.$$

So using the inequality (6), $V(x) - \frac{x}{\delta} \leq \frac{p}{\delta^2}$ and $V(x) - \left(\frac{x}{\delta} + \frac{p}{\delta^2}\right) \geq -\frac{p}{\delta^2}$. Therefore

$$g_1(x) - \frac{x}{\delta} = V(x) - \frac{x}{\delta} - h_1(x) \leq \frac{p}{\delta^2} - h_1(0) - \frac{\delta h_1(0)}{p} x,$$

and

$$\begin{aligned} g_2(x) - \left(\frac{x}{\delta} + \frac{p}{\delta^2}\right) &= V(x) - \left(\frac{x}{\delta} + \frac{p}{\delta^2}\right) + h_2(x) \\ &\geq -\frac{p}{\delta^2} + h_2(0) + \frac{\delta h_2(0)}{p} x. \end{aligned}$$

Therefore if $x > \frac{p/\delta^2 - h_1(0)}{\delta h_1(0)/p}$ then, $g_1(x) < \frac{x}{\delta}$ and if $x > \frac{p/\delta^2 - h_2(0)}{\delta h_2(0)/p}$, then $g_2(x) > \frac{x}{\delta} + \frac{p}{\delta^2}$. □

Lemma 4 *If* $V(x)$ *has a Radon–Nikodym derivative* $v(x)$ *such that* $\lim_{x\to\infty} v(x)$ *exists, then* $\lim_{x\to\infty} v(x) = 1/\delta$.

Proof Since $V(x)$ is absolutely continuous, by the Radon–Nikodym theorem, there exists a bounded Lebesgue integrable function $v(x)$ such that

$$V(x) = V(0) + \int_0^x v(u)\,\mathrm{d}u.$$

Therefore $v(x) < K$ and assume $L := \lim_{x\to\infty} v(x)$. Let $\epsilon > 0$, we can find x_0 such that if $x > x_0$ then $|v(x) - L| < \epsilon/3$. Assuming x large enough we also have $\frac{V(0)+Kx_0}{x} < \epsilon/3$. Therefore

$$\begin{aligned}\left|\frac{V(x)}{x} - v(x)\right| &= \frac{V(0)}{x} + \frac{1}{x}\int_0^{x_0} |v(u) - v(x)|\,\mathrm{d}u + \frac{1}{x}\int_{x_0}^{x} |v(u) - v(x)|\,\mathrm{d}u \\ &\le \frac{V(0) + Kx_0}{x} + \frac{1}{x}\int_{x_0}^{x} (|v(u) - L| + |L - v(x)|)\,\mathrm{d}u \\ &\le \epsilon/3 + 2\frac{x - x_0}{x}\epsilon/3 < \epsilon.\end{aligned}$$

Therefore $\lim_{x\to\infty} v(x) = \lim_{x\to\infty} \frac{V(x)}{x}$. But by (6) we know that $\lim_{x\to\infty} \frac{V(x)}{x} = 1/\delta$. □

Remark 2 If we assume that $V(x)$ is a concave function , then $V(x)$ has a decreasing Radon–Nikodym derivative and therefore Lemma 4 holds.

Remark 3 Note that even if $V(x)$ is differentiable, from $\lim_{x\to\infty} \frac{V(x)}{x} = 1/\delta$, we can not reach the conclusion that $\lim_{x\to\infty} V'(x) = 1/\delta$. For example consider differentiable, Lipschitz continuous and strictly increasing function $g(x) = a + \frac{x}{\delta} + \frac{\sin x}{2\delta}$. We have $g(x)/x \to 1/\delta$ but $\lim_{x\to\infty} g'(x)$ does not exists.

The next Theorem is important from a practical point of view. In this theorem we prove that at least in lines with proportional reinsurance, for large x, the optimal strategy is such that there is no need for reinsurance.

Theorem 2 *If* Y_{ijk} *has probability density function* $f_{ij}(x)$ *with* $E(Y_{ijk}) < \infty$ *and the line* ℓ *is proportional i.e.* $r_\ell(u, y) = uy$, *and* $V(x)$ *has a Radon–Nikodym derivative* $v(x)$ *with a limit at infinity, then there exists* $x_1 > 0$ *such that if* $x > x_1$ *then* $H_V(x, \boldsymbol{u})$ *defined in* (7) *is an increasing function with respect to* u_ℓ *on* $(\underline{u_\ell}, 1]$ *for every* $0 < \underline{u_\ell} < 1$.

Proof Since $V(x)$ is absolutely continuous, $H_V(x, \boldsymbol{u})$ is a.e. differentiable with respect to u_ℓ. We show that $\frac{\partial}{\partial u_\ell} H_V(x, \boldsymbol{u}) > 0$ a.e. $u_\ell \in (\underline{u_\ell}, 1]$. It is easy to see that

$$\frac{\partial}{\partial u_\ell} p(\boldsymbol{u}) = (1+\theta_\ell) \sum_{i \in \mathcal{I}_\ell} \beta_i \mu_{i\ell} > 0,$$

and (almost everywhere)

$$\frac{\partial}{\partial u_\ell} \sum_{i=1}^{m} \beta_i \int_0^x V(x-z)\, \mathrm{d}F_i^{\boldsymbol{u}}(z) = \sum_{i \in \mathcal{I}_\ell} \beta_i \frac{\partial}{\partial u_\ell} \int_0^x V(x-z)\, \mathrm{d}F_i^{\boldsymbol{u}}(z),$$

where $\mathcal{I}_\ell = \{ i \in \{1,2,\dots,m\} | \ell \in A_i \}$. Fix $i \in \mathcal{I}_\ell$ and define $S = \sum_{j \in A_i} u_j Y_{ij}$ and $S_\ell = \sum_{j \in A_i, j \neq \ell} u_j Y_{ij}$. Since $E(Y_{ij}) < \infty$, $\lim_{x \to \infty} x f_{Y_{ij}}(x) = 0$ and by the dominated convergence theorem we have $\lim_{x\to\infty} E\big(Y_{ij} I_{\{S>x\}}\big) = 0$. Therefore by (4), for $\epsilon > 0$ we can find x_0 such that if $x > x_0$ we have

$$x f_{Y_{ij}}(x) < \epsilon, \quad |v(x) - 1/\delta| < \epsilon,$$

and $x_1 > x_0$, such that if $x > x_1$ we have

$$E\big(Y_{ij} I_{\{S > x - x_0\}}\big) < \epsilon.$$

Note that $\int_0^x V(x-z)\, \mathrm{d}F_i^{\boldsymbol{u}}(z) = E\big[V(x-S) I_{\{S \le x\}}\big]$, by conditioning on S_ℓ we have

$$\begin{aligned} E\big[V(x-S) I_{\{S\le x\}}\big] &= \int_0^\infty E\Big[V\big(x - S_\ell - u_\ell Y_{i\ell}\big) I_{\{S_\ell + u_\ell Y_{i\ell} \le x\}} | S_\ell = z\Big] f_{S_\ell}(z) \mathrm{d}z \\ &= \int_0^x E\Big[V\big(x - z - u_\ell Y_{i\ell}\big) I_{\{u_\ell Y_{i\ell} \le x - z\}}\Big] f_{S_\ell}(z) \mathrm{d}z \\ &= \int_0^x \int_0^{\frac{x-z}{u_\ell}} V\big(x - z - u_\ell y\big) f_{Y_{i\ell}}(y)\, \mathrm{d}y f_{S_\ell}(z) \mathrm{d}z. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial u_\ell} \int_0^x V(x-z)\, \mathrm{d}F_i^{\boldsymbol{u}}(z) = &- \int_0^x \left[\frac{x-z}{u_\ell^2} V(0) f_{Y_{i\ell}}\left(\frac{x-z}{u_\ell} \right) \right] f_{S_\ell}(z) \mathrm{d}z \\ &- \int_0^x \int_0^{\frac{x-z}{u_\ell}} y v\big(x - z - u_\ell y\big) f_{Y_{i\ell}}(y)\, \mathrm{d}y f_{S_\ell}(z) \mathrm{d}z \\ = &-C_1 - C_2. \end{aligned}$$

Note that $\int_0^x f_{Y_{i\ell}}\Big(\frac{x-z}{u_\ell}\Big) f_{S_\ell}(z)\mathrm{d}z = u_\ell f_S(x)$ and $\int_0^x z f_{Y_{i\ell}}\Big(\frac{x-z}{u_\ell}\Big) f_{S_\ell}(z)\mathrm{d}z \ge 0$ and therefore

$$C_1 \le \frac{x V(0)}{u_\ell} f_S(x) < \frac{V(0)\epsilon}{u_\ell}$$

and

$$\begin{aligned}
C_2 &= \int_0^x \int_0^{\frac{x-z}{u_\ell}} yv\big(x-z-u_\ell y\big) I_{\{z+u_\ell y \le x-x_0\}} f_{Y_{i\ell}}(y)\,\mathrm{d}y f_{S_\ell}(z)\mathrm{d}z \\
&\quad + \int_0^x \int_0^{\frac{x-z}{u_\ell}} yv\big(x-z-u_\ell y\big) I_{\{z+u_\ell y > x-x_0\}} f_{Y_{i\ell}}(y)\,\mathrm{d}y f_{S_\ell}(z)\mathrm{d}z \\
&\le \int_0^x \int_0^{\frac{x-z}{u_\ell}} y\Big(\frac{1}{\delta}+\epsilon\Big) I_{\{z+u_\ell y \le x-x_0\}} f_{Y_{i\ell}}(y)\,\mathrm{d}y f_{S_\ell}(z)\mathrm{d}z \\
&\quad + KE\big(Y_{i\ell} I_{\{S > x-x_0\}}\big) \\
&\le \Big(\frac{1}{\delta}+\epsilon\Big)\mu_{i\ell} + KE\big(Y_{i\ell} I_{\{S>x-x_0\}}\big).
\end{aligned}$$

So if $x > x_1$,

$$\begin{aligned}
\frac{\partial}{\partial u_\ell} H_V(x,\boldsymbol{u}) &= \frac{\partial}{\partial u_\ell} p(\boldsymbol{u}) v(x) + \frac{\partial}{\partial u_\ell} \sum_{i=1}^m \beta_i \int_0^x V(x-z)\,\mathrm{d}F_i^{\boldsymbol{u}}(z) \\
&\ge (1+\theta_\ell) \sum_{i\in\mathcal{I}_\ell} \beta_i \mu_{i\ell}\Big(\frac{1}{\delta}-\epsilon\Big) - C_1 - C_2 \\
&\ge \frac{\partial}{\partial u_\ell} p(\boldsymbol{u}) v(x) - \sum_{i\in\mathcal{I}_\ell} \Big[\beta_i V(0)\epsilon/u_\ell - \beta_i\Big(\frac{1}{\delta}+\epsilon\Big)E(Y_{i\ell}) - K\epsilon\Big] \\
&\ge \sum_{i\in\mathcal{I}_\ell} \Big[(1+\theta_\ell)\beta_i\mu_{i\ell}\Big(\frac{1}{\delta}-\epsilon\Big) - \beta_i V(0)\epsilon/u_\ell - \beta_i\Big(\frac{1}{\delta}+\epsilon\Big)\mu_{i\ell} - K\epsilon\Big] \\
&\ge \sum_{i\in\mathcal{I}_\ell} \Big[\frac{\theta_\ell \beta_i \mu_{i\ell}}{\delta}\Big] - \sum_{i\in\mathcal{I}_\ell} \Big[(1+\theta_\ell)\beta_i\mu_{i\ell} + \beta_i V(0)/\underline{u_\ell} + \beta_i\mu_{i\ell} + K\Big]\epsilon.
\end{aligned}$$

Therefore if ϵ is small enough, then for almost all $u_\ell \in (\underline{u_\ell}, 1]$, $\frac{\partial}{\partial u_\ell} H_V(x,\boldsymbol{u}) > 0$. □

Corollary 1 *Under assumptions of Theorem 2, there exist x_0 such that if $x > x_0$ then*

$$\sup_{p_j(r_j)>0} H(x,u_1,u_2,\dots,u_n) = H(x,1,1,\dots,1).$$

4 Numerical solution and examples

To find the optimal value function, we must first find $V(0)$. Using Theorem 1 and Lemma 4, we have provided a method to numerically obtain a solution of (8) that satisfies the inequality (6), and therefore by Theorem 1, it is the value function $V(x)$.

Consider small numbers $\Delta x > 0$, $e > 0$ and positive integer m. By inequality (6) we have $V(0) \in \Big[0, \frac{p}{\delta^2}\Big]$. Using finite difference method we find a solution $g_1(x)$ of (8) with $g_1(0) = \frac{0+\frac{p}{\delta^2}}{2} = \frac{p}{2\delta^2}$:

$$g_1(0) = \frac{p}{2\delta^2}, \qquad \mathcal{D}g_1(0) = \inf_{\{\boldsymbol{u}\in\mathcal{U}|p(\boldsymbol{u})>0\}} \frac{(\delta+\beta)g_1(0)}{p(\boldsymbol{u})},$$
$$g_1((i+1)\Delta x) \approx g_1(i\Delta x) + \mathcal{D}g_1(i\Delta x)\Delta x, \quad i = 0, 1, \ldots. \tag{11}$$

Therefore we can approximate a solution of (8) at $x = 0, \Delta x, 2\Delta x, \ldots$. We continue increasing i until one of the following conditions occurs:

1. $g_1(i^*\Delta x) < \frac{i^*\Delta x}{\delta}$ and $\frac{i\Delta x}{\delta} \leq g_1(i\Delta x) \leq \frac{i\Delta x}{\delta} + \frac{p}{\delta^2}$ for $i \in \{0, 1, \ldots i^* - 1\}$.
2. $g_1(i^*\Delta x) > \frac{i^*\Delta x}{\delta} + \frac{p}{\delta^2}$ and $\frac{i\Delta x}{\delta} \leq g_1(i\Delta x) \leq \frac{i\Delta x}{\delta} + \frac{p}{\delta^2}$ for $i \in \{0, 1, \ldots i^* - 1\}$.
3. $\left|\mathcal{D}g_1(i\Delta x) - \frac{1}{\delta}\right| < e$ for $i \in \{i^*, i^*+1, \ldots i^*+m\}$, and $\frac{i\Delta x}{\delta} \leq g_1(i\Delta x) \leq \frac{i\Delta x}{\delta} + \frac{p}{\delta^2}$ for $i \in \{0, 1, \ldots i^*+m\}$.

If condition 3 occurs, we accept $g_1(x)$ as an approximation of the value function. If condition 1 occurs, from Theorem 1 we conclude that $g_1(0) < V(0)$ and therefore $V(0) \in \left[\frac{p}{2\delta^2}, \frac{p}{\delta^2}\right]$; thus, we must find another solution $g_2(x)$ with $g_2(0) = \frac{\frac{p}{2\delta^2}+\frac{p}{\delta^2}}{2}$. On the other hand if condition 2 occurs, $g_1(0) > V(0)$ and therefore $V(0) \in \left[0, \frac{p}{2\delta^2}\right]$; hence, we must find another solution $g_2(x)$ with $g_2(0) = \frac{0+\frac{p}{2\delta^2}}{2}$.

By repeating the above steps we can identify an approximation for the value function. In the following example, we explain this numerical algorithm.

Algorithm 1 Bisection algorithm for finding $V(x)$ numerically, based on Theorem 1.

Consider $a = 0$, $b = p/\delta^2$ and Convergence=False
while Convergence is False **do**
  $c = (a+b)/2$
  Find the unique solution $g(x)$ of (7) with $g(0) = c$.
  if $\frac{x}{\delta} < g(x) \leq \frac{x}{\delta} + \frac{p}{\delta^2}$ **then**
    Convergence = True
  else if $g(x) < \frac{x}{\delta}$ for large x. **then**
    $a \leftarrow (a+b)/2$
  else if $g(x) > \frac{x}{\delta} + \frac{p}{\delta^2}$ for large x. **then**
    $b \leftarrow (a+b)/2$
  end if
end while
$V(x) = g(x)$.

Example 1 Consider a company that operates on two lines with three sources of claims $A_1 = \{1\}$, $A_2 = \{2\}$ and $A_3 = \{1,2\}$. Using the model presented in relation (3), the capital process can be written as follows.

$$X(t) = x + \int_0^t p(U_1(s), U_2(s))\mathrm{d}s - \sum_{k=1}^{N_1(t)} r_1\big(U_1(\tau_{1k}^-), Y_{1k}\big)$$
$$- \sum_{k=1}^{N_2(t)} r_2\big(U_2(\tau_{2k}^-), Y_{2k}\big) - \sum_{k=1}^{N_3(t)} \big[r_1\big(U_1(\tau_{3k}^-), Y_{31k}\big) + r_2\big(U_2(\tau_{3k}^-), Y_{32k}\big)\big].$$

Assume light tail distribution $\Gamma(3, 1.25)$ for Y_{1k}, heavy tail distribution *Weibull*(2, 0.5) for Y_{2k}, $Y_{31k} \sim Exp(0.25)$, $Y_{32k} \sim Exp(0.5)$, $\beta_1 = 5$, $\beta_2 = 6$, $\beta_3 = 4$, $\eta_1 = \eta_2 = 0.25$, $\theta_1 = \theta_2 = 0.3$ and $\delta = 0.15$. We apply the proportional strategy to the first and the excess of loss strategy to the second line, respectively:

$$r_1(y, u) = uy, \qquad r_2(y, u) = \min(u, y).$$

It is easy to see that the distribution of $u_1 Y_{31k} + \min(Y_{32k}, u_2)$ is

$$F_3^{(u_1,u_2)}(x) = 1 - \begin{cases} \frac{2u_1 e^{-\frac{x}{4u_1}} - e^{-\frac{x}{2}}}{2u_1 - 1} I_{(-\infty,u_2)}(x) + \frac{2u_1 - e^{\frac{u_2}{4u_1} - \frac{u_2}{2}}}{2u_1 - 1} e^{-\frac{x}{4u_1}} I_{[u_2,\infty)}(x) & u_1 \neq \frac{1}{2} \\ \frac{x+2}{2} e^{-\frac{x}{2}} I_{(-\infty,u_2)}(x) + \frac{u_2+2}{2} e^{-\frac{x}{2}} I_{[u_2,\infty)}(x) & u_1 = \frac{1}{2} \end{cases}.$$

Now, we must find a solution of (8) that satisfies the inequality (6). From inequality (6), we have $0 < V(0) < p/\delta^2 = 5333.3$. By solving (8) with initial value $g(0) = (0 + 5333.3)/2 = 2666.7$, we have $g(x) > x/\delta + p/\delta^2$ for large x. Thus, by Theorem 1, $0 < V(0) < 2666.7$. Therefore, we choose 2666.7 as the new upper bound for $V(0)$ and repeat the above procedure until the inequality (6) is met. Note that we can only check (6) on $[0, x_M]$ for some large x_M. To overcome this problem, we have used condition 3 and therefore $x_M = (i^* + m)\Delta x$.

The steps of the Algorithm 1 are available in Table 1. Also the value function and all iterations of the Algorithm 1 are presented in Fig. 2.

In Fig. 3 we can see the optimal reinsurance strategies. As expected from Theorem 2, in Fig. 3a, for $x > 22.8$, we have $u_1^*(x) = 1$.

Table 1 Bisection steps for finding $V(0)$

Iteration	Bound for $V(0)$	$g(0)$	$g(x)$ for large x
1	$(0000.0, 5333.3)$	$g_1(0) = 2666.7$	$g_1(x) > x/\delta + p/\delta^2$
2	$(0000.0, 2666.6)$	$g_2(0) = 1333.3$	$g_2(x) > x/\delta + p/\delta^2$
3	$(0000.0, 1333.3)$	$g_3(0) = 666.67$	$g_3(x) > x/\delta + p/\delta^2$
4	$(0000.0, 666.66)$	$g_4(0) = 333.33$	$g_4(x) > x/\delta + p/\delta^2$
5	$(0000.0, 333.33)$	$g_5(0) = 166.67$	$g_5(x) < x/\delta$
6	$(166.66, 333.33)$	$g_6(0) = 250.00$	$g_6(x) > x/\delta + p/\delta^2$
7	$(166.66, 250.00)$	$g_7(0) = 208.33$	$g_7(x) < x/\delta$
8	$(208.33, 250.00)$	$g_8(0) = 229.17$	$g_8(x) > x/\delta + p/\delta^2$
9	$(208.33, 229.17)$	$g_9(0) = 218.75$	$g_9(x)$ satisfies (6)

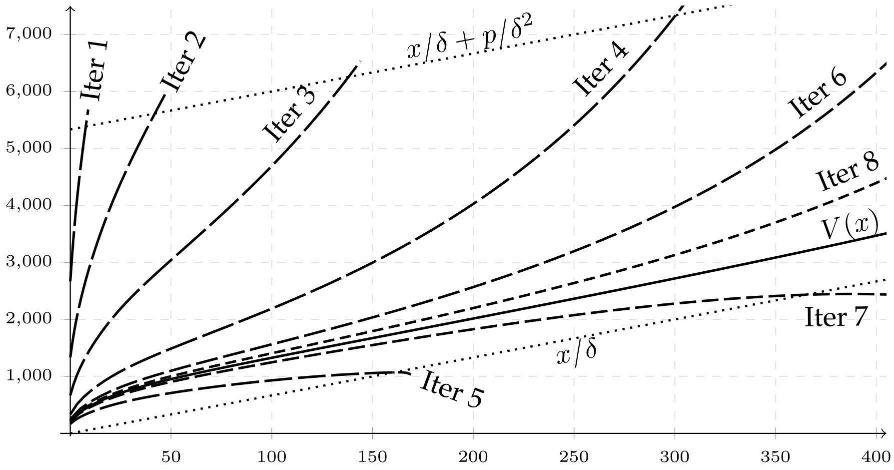


Fig. 2 Plot of $g(x)$ for different $g(0)$ in Table 1

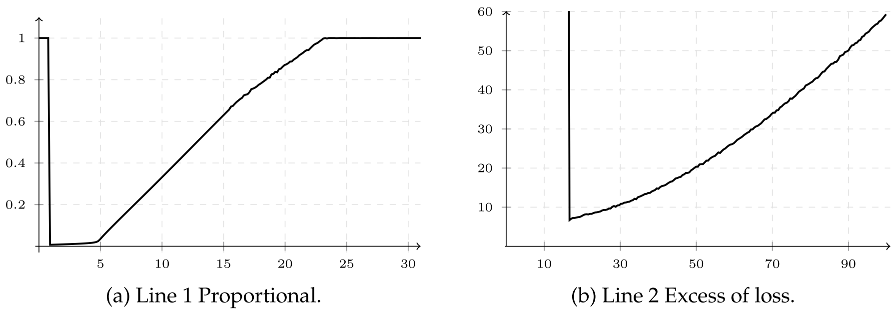


Fig. 3 Optimal reinsurance strategies in Example 1

Table 2 Combinations of contract types

	Line 1	Line 2	Value function
1	Excess of loss	Excess of loss	$V_{XL,XL}(x)$
2	Proportional	Excess of loss	$V_{P,XL}(x)$ (Example 1)
3	Excess of loss	Proportional	$V_{XL,P}(x)$
4	Proportional	Proportional	$V_{P,P}(x)$

To examine the impact of different contracts in different lines, we repeat Example 1 for the other combinations of contract types (Table 2).

Figure 4, has displayed functions $V_{XL,XL}(x) - V_{P,XL}(x)$, $V_{XL,XL}(x) - V_{XL,P}(x)$ and $V_{XL,XL}(x) - V_{P,P}(x)$. As a result, the best combination is the excess of loss contract in both lines. Moreover, We have $V_{P,XL}(x) > V_{XL,P}(x)$. In Example 1, the second line has a heavy tail distribution for claim sizes, therefore it can be argued that it

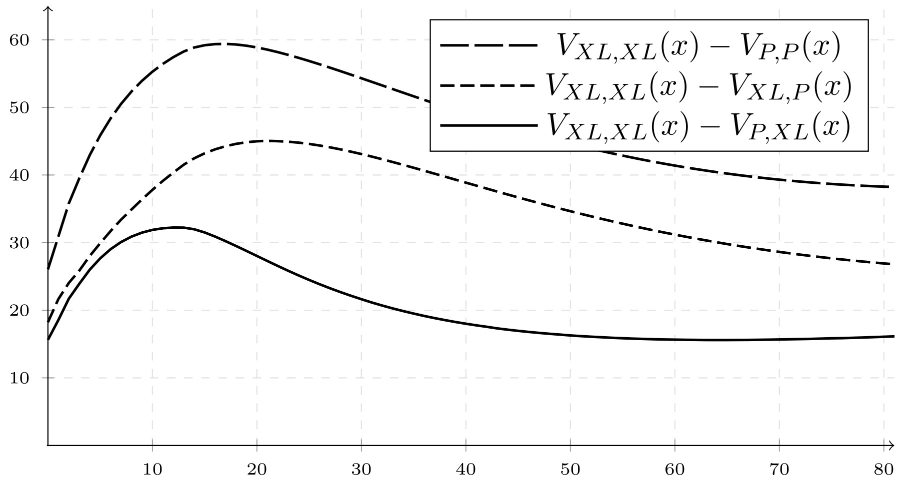


Fig. 4 Impact of different contract types in Example 1

is recommended to apply the excess of loss contracts, at least in lines which have heavy tail distributions.

To be able to compare the objective function used in this paper with the survival probability, in the next example, we have used the same settings as Example 2 of Masoumifard and Zokaei [9]. Since claims can not occur simultaneously in Masoumifard and Zokaei [9], we assume $A_1 = \{1\}$, $A_2 = \{2\}$ and $A_3 = \{3\}$.

Example 2 Assume a company with three sources and three lines such that $A_1 = \{1\}$, $A_2 = \{2\}$ and $A_3 = \{3\}$. Consider the distribution $F_1(x) = 1 - e^{-0.5x}$ for the claim sizes in the first line, $F_2(x) = 1 - \left(\frac{3}{x+3}\right)^3$ for the claim sizes in the second line and the mixture $F_3(x) = 0.7F_1(x) + 0.3F_2(x)$ for the claim sizes in the third line. Let $\beta_1 = 8$, $\beta_2 = 4$, $\beta_3 = 5$, $\eta_1 = 0.3$, $\theta_1 = 0.35$, $\eta_2 = 0.2$, $\theta_2 = 0.25$, $\eta_3 = 0.25$, $\theta_3 = 0.3$ and $\delta = 0.005, 0.01, 0.05, 0.1, 0.2$.

Using proportional reinsurance for all lines, we have solved the problem for two objective functions:

- Survival Probability $S_U(x) = 1 - P(\tau_U < \infty | X_U(0) = x)$, which is used in Masoumifard and Zokaei [9] (denoted by $\delta_U(x)$). The optimal survival probability is $S(x) = \sup_U S_U(x)$
- The value function used in this paper (defined by (4)).

We have calculated the followings:

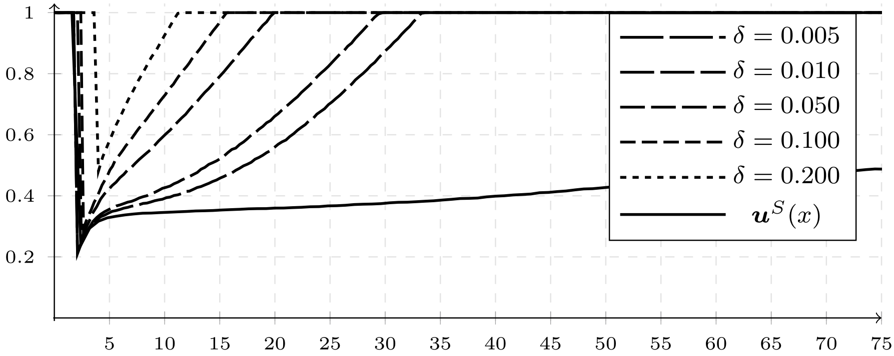
- Optimal survival probability $S(x)$ and its corresponding optimal strategy denoted by $\boldsymbol{U}^S(t) = \boldsymbol{u}^S(X_{\boldsymbol{U}^S}(t^-))$.
- The value function $V(x)$ and its corresponding optimal strategy denoted by $\boldsymbol{U}^*(t) = \boldsymbol{u}^*(X_{\boldsymbol{U}^*}(t^-))$.
- The function (4) for reinsurance strategy $\boldsymbol{U}^S$, i.e. $V_{\boldsymbol{U}^S}(x)$.
- Survival probability for our optimal strategy $\boldsymbol{U}^*$, i.e $S_{\boldsymbol{U}^*}(x)$.

Using objective function (4) instead of the survival probability gives us a better strategy from a practical point of view. That is, considering the survival probability as objective function, even for large x , a significant amount of reinsurance must be purchased (see Fig. 5). This results in a large amount of surplus being paid to the reinsurer in exchange for a slight increase in the probability of survival. However, according to Theorem 2 and Fig. 5, we see that in all cases adopting (4) as objective function leads to strategies that $u^*(x) = 1$ for large x and for larger δ, the optimal strategy tends to 1 faster. In other words, there is no need for reinsurance when the surplus is large enough.

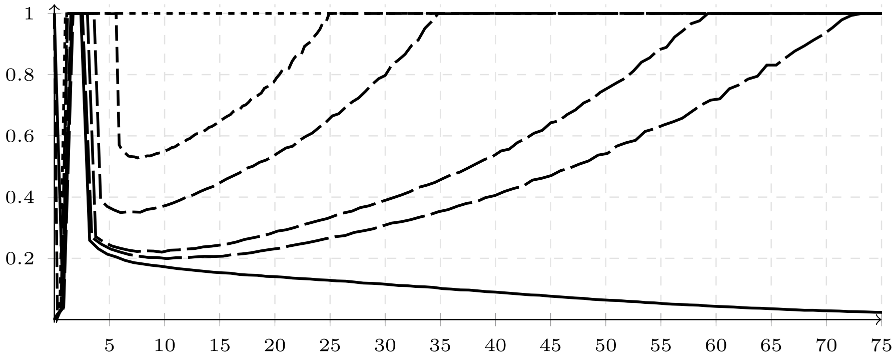
In Fig. 6 we can see a large difference between $V(x)$ and $V_{\boldsymbol{U}^S}(x)$ and very small difference between $S(x)$ and $S_{\boldsymbol{U}^*}(x)$ for small δ. On the other hand for large δ, the difference between $S(x)$ and $S_{\boldsymbol{U}^*}(x)$ becomes larger. So using objective function (4) with small δ can have some advantages over survival probability.

5 Conclusion

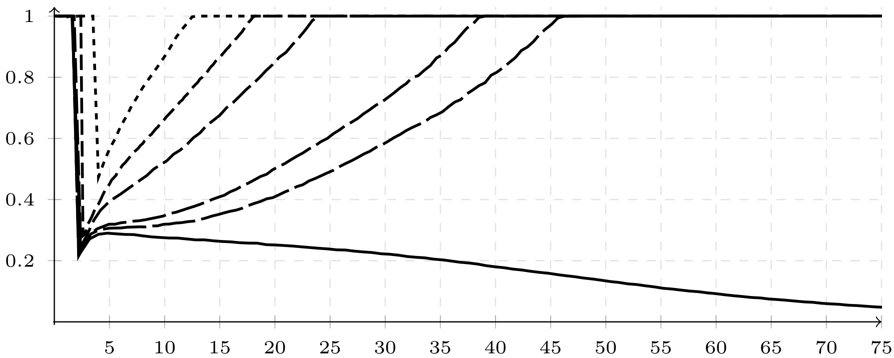
In this paper, we have considered an insurance company that is active in multiple lines as such that claims can occur simultaneously on several lines. A vector of dynamic reinsurance strategies has been derived as such that the expected discounted surplus level integrated over time (objective function) is maximized. Using Theorem 1, we characterized the optimal objective function as the unique solution of the associated HJB equation that satisfies inequality (6). We also presented an algorithmic method for finding the value function numerically. Moreover, by comparing the objective function used in this paper with the survival probability, it is safe to conclude that using this objective function may have advantages over the survival probability.



(a) Line 1.

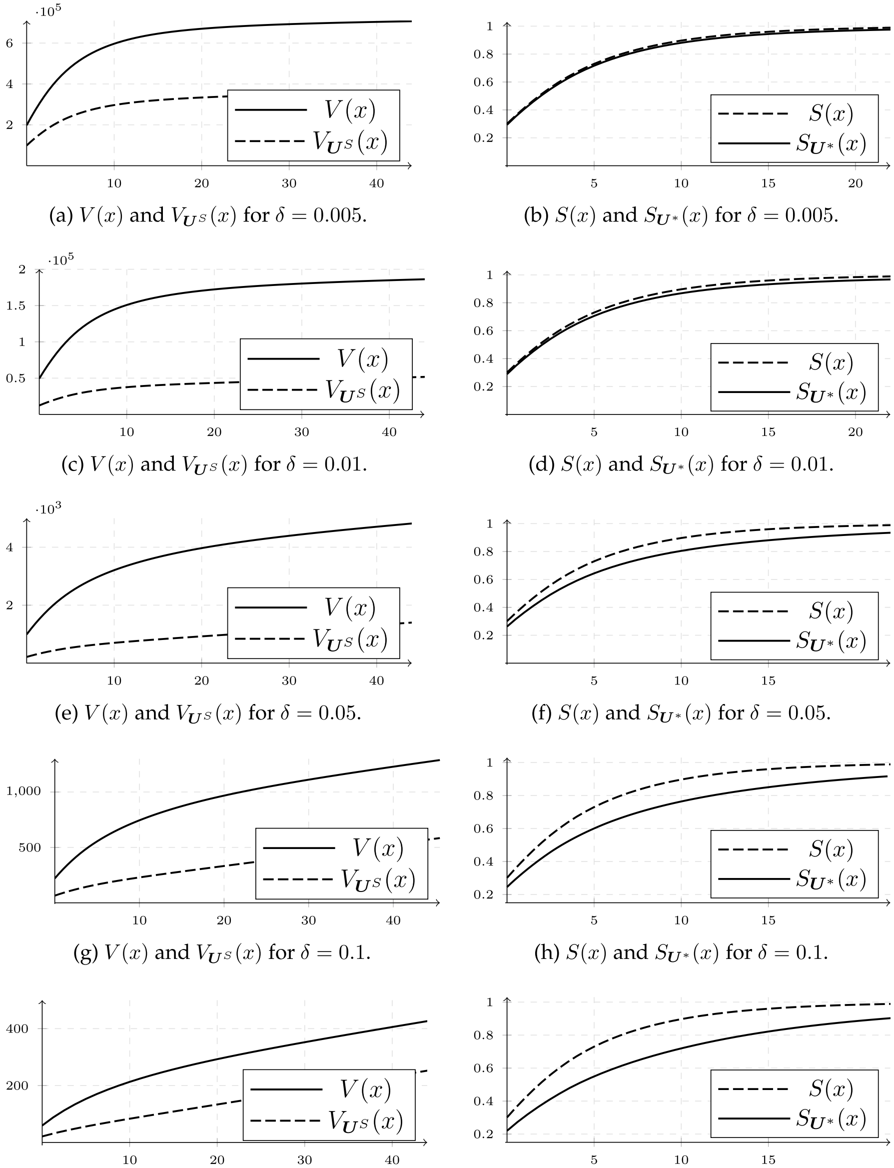


(b) Line 2.



(c) Line 3.

Fig. 5 Optimal strategies $\boldsymbol{u}^*(x)$ for $\delta = 0.005,\ 0.01,\ 0.05,\ 0.1,\ 0.2$ and $\boldsymbol{u}^S(x)$ in Example 2



(a) $V(x)$ and $V_{U^S}(x)$ for $\delta = 0.005$.
(b) $S(x)$ and $S_{U^*}(x)$ for $\delta = 0.005$.
(c) $V(x)$ and $V_{U^S}(x)$ for $\delta = 0.01$.
(d) $S(x)$ and $S_{U^*}(x)$ for $\delta = 0.01$.
(e) $V(x)$ and $V_{U^S}(x)$ for $\delta = 0.05$.
(f) $S(x)$ and $S_{U^*}(x)$ for $\delta = 0.05$.
(g) $V(x)$ and $V_{U^S}(x)$ for $\delta = 0.1$.
(h) $S(x)$ and $S_{U^*}(x)$ for $\delta = 0.1$.
(i) $V(x)$ and $V_{U^S}(x)$ for $\delta = 0.2$.
(j) $S(x)$ and $S_{U^*}(x)$ for $\delta = 0.2$.

Fig. 6 Comparison of the two value functions for various values of δ

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